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Analytical Study of Non-Normal Classes of Operators

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***Abstract*-- This paper presents the study of Non-normal Classes of operators, in which we discuss about quasi-normal , subnormal, hyponormal, seminormal, paranormal, i.e., class of (N), operators of class (N, K) and normaloid operators. Here , it is proved in this paper that Normal operator ⊂ quasi- normal operator ⊂ Subnormal operator ⊂ hyponormal operator ⊂ operators of class (N)⊂operators of Class (N,k) ⊂ normaloid operators.**

***Keywords*-- Hilbert space, Bounded linear operators, NormalOperator, Isometry, Kernel of homomorphism**

I. INTRODUCTION

Beals, R. (1), and Ghosh, H.C. (2,3) are the pioneer workers of the present area. Infact, the present work is the extension of work done by Istratescu , V.I.(2), Yosida, K. (15), Srivastava et al. (4), Srivastava et al. (5), Srivastava et al. (6), Srivastava et al. (7), Kumar et al. (8), Srivastava et al. (9), Kumar et al. (10), and Srivastava et al. (11). In this paper, we have studied analytically non- normal classes of operators.

Here, we use the following definitions and fundamental ideas :

Let H be a Hilbert space and L (H) be the set of all bounded linear operators on H.

DEFINITION 1 :- An element $T \in L(H)$ is called quasi – normal if

$$(T^*T)T = T(T^*T)$$

DEFINITION 2 :- An element $T \in L(H)$ is called subnormal if there exists a

Hilbert space $K \supset H$, H is a closed subspace of K, and a normaloperator $N \in L(K)$ such that $Nx = Tx$, for all $x \in H$

DEFINITION 3 :- An element $T \in L(H)$ is called hyponormal if for all $x \in H$,

$\| T^* x \| \leq \| Tx \|$. If T* is hyponormal , we say T is seminormal .

DEFINITION 4 :- An element $T \in L(H)$ is called of class (N) or paranormal if for all $x \in H$, $\| x \| = 1$, $\| Tx \|^2 \leq \| T^2x \|$.

DEFINITION 5 :- An element $T \in L(H)$ is called of class (N, K) if for all $x \in H$,$\| x \| = 1$, $\| Tx \|^k \leq \| T^k x \|$

DEFINITION 6 :- An element $T \in L(H)$ is called normaloid if $r_T = \| T \|$.

The following gives the most important connections between the classes considered above, which is the *Main result as:-*

Normal operator ⊂ quasi – normal operator ⊂ subnormal operator ⊂ hyponormal operator ⊂ operators of class (N) ⊂ operators of class (N, K) ⊂ normaloid operators.

PROOF :- It is obvious that every normal operator is quasi – normal .We give now an example of a quasi – normal operator which is not normal .It is clear that every isometry is such an operator.We prove now that every quasi- normal operator is subnormal ,

First, we remark that if T is quasi – normal , then

$$\ker T = \{x \mid Tx = 0\}$$

is a reducing subspace , i.e., invariant for T and T* . Indeed, if $x \in \ker T$, we have $T^*T = 0$, and since T is quasi – normal, $T^* x = 0$. Thus we can decompose T, $T = 0 \oplus T_1$

where T_1 is also quasi – normal , but $\ker T_1 = \{0\}$. Let $T = UR$ and thus U is an isometry, and $UR = RU$ gives $U^*R = RU^*$. If $E = UU^*$ then $(I - E)U = U^*(I - E) = 0$.

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The normal extension of T is now constructed as follows: We consider theHilbert space H = H ⊕ H and operators

$$T_1 = \begin{bmatrix} U & I - E \\ 0 & U^* \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$$

Acting on H . Clearly the operator T_2 is positive, and it is easy to see that T_1 is an isometric operator. Also, a simple computation shows that $T_1\ T_2 = T_2\ T_1$. The normal operator is defined as

$$N = \begin{bmatrix} T_1 & (I - E)R \\ 0 & U^* R \end{bmatrix}$$

and it is obviously an extension of T.

We give now an example of an operator which is subnormal and not quasi – normal . First we remark that the "unilateral shift"

$Te_i = e_{i+1}$ $\qquad$ i = 1, 2, 3, …………

on the space $\ell_+^2 = \{X | \Sigma_{i=0}^{\infty} |X_i|^{\,2} < \infty\}$ is subnormal . Indeed, we consider the space

$\ell^2 = \{X | \Sigma_{i=0}^{\infty} |X_i|^{\,2} < \infty\}$ and the bilateral shift

$Te_i = e_{i+1}$ $\quad$ i = …… -1, 0, 1, 2, ………..

which is unitary and is an extension of T.

It is also obvious that if T is subnormal, then T + z is also subnormal for all z .

We prove now that every subnormal operator is hyponormal. Indeed, if P is theprojection

P : K → H , then we have for all x , y ϵ H ,

< T* , x , y > = < x , Ty > = < x , Ny > = < N*x , y > = < N*x , Py > = < PN*x , y >and thus the operator PN*has H as an Invariant subspace . Thus

$$T^*x = PN^*x .$$

We obtain that

$$\| T^*x \| = \| PN^*x \| \leq \| N^*x \| = \| Nx \| = \| Tx \|$$

and the assertion is proved .

We give now an example of an operator which is hyponormal and not subnormal. First we give a necessity condition for subnormality of an operator : If T ϵ L (H) is subnormal , then for any $x_0 , x_1 , \ldots , x_n$ ϵ H ,

$$\sum_{j,i} < T^i x_j , T^j x_i > c^{*}\ c_i \geqslant 0,$$

where { c_j} are the complex numbers.Indeed, for T is subnormal , we have

$$\| \Sigma N^{*j} x_j \|^2 = < \Sigma N^{*j} x_j , \Sigma N^{*i} x_i > = \Sigma_j \Sigma_i < N^{*j} x_j , N^{*i} x_i >$$

$$= \Sigma_j \Sigma_i < N^i N^{*j} x_j , x_i > = \Sigma_j \Sigma_i < N^{*j} N^i x_j , x_i >$$

$$= \Sigma_j \Sigma_i < N^i x_j , N^j x_i > = \Sigma_j \Sigma_i < T^i x_j , T^j x_i >$$

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and if we consider $x_i = c_i x_i$, we obtain the assertion.

Now on any separable Hilbert space with the orthonormalbasis $\{ e_i^{\infty} \}_{i=1}$ we consider the operator

$$Te = \alpha_i e_{i+1}$$

Where $\{\alpha_i\}$ is a bounded sequence of complex numbers . This operator is the so-called weighted shift with the weighted sequence $\{\alpha_i\}$.

If $|\alpha_i|<|\alpha_{i+1}|$, then it is easy to see that T is hyponormal . If $\alpha_0 = \alpha \alpha_1 = \beta , \alpha_2 = \alpha_3 = \ldots = 1$

With $\alpha < \beta$, then we have a hyponormal operator which is not subnormal sincefor i , j = 0 , 1 , 2, and $x_0 = e_0$, $x_1 = e_1$, $x_2 = e_2$, the corresponding matrix is

$$M = \begin{bmatrix} 1 & \alpha & \alpha\beta \\ \alpha & \beta & \beta \\ \alpha\beta & \beta & 1 \end{bmatrix}$$

and det $M < 0$.

We prove now that every hyponormal operator is of class (N) . Indeed , for any$x \in H$,$\| x \| = 1$,

$$\| Tx \|^2 = < Tx , Ty > \leq < T^*Tx , x > \leq \| T^2 x \|$$

To prove that the class of operators with the property that all x , $\| x \| = 1$, $\| Tx \|^2 \leq \| T^2 x \|$, is larger than the class of hyponormal operators , we prove first a result about operators of this class : If T is of class (N) , then T^2 isalso of class (N) . Indeed , for any x, $\| x \| = 1$, we have

$$\|T^4 x\| = \left\| \frac{T^2T^2X}{\|T^2X\|} \right\| \; \|T^2X\| \geq \frac{\|T^3X\|^2}{\|T^2X\|}$$

and since

$$\|T^3 x\| = \left\| \frac{T^2TX}{\|TX\|} \right\| \; \|TX\| \geq \frac{\|T^2X\|^2}{\|TX\|}$$

which gives

$$\| T^4 x\| \geqslant \|T^2 x\|^2 \; \| T x \|^2 \|T^2 x\| = \|T^2 x\|^2 \frac{\|T2x\|}{\|T x\|^2} \geqslant \|T^2 x\|^2$$

and the assertion is proved .

For an example of an operator which is of class (N) and not hyponormal,we givean example of a hyponormal operator T for which T^2 is not a hyponormal operator .Let H be a Hilbert space and consider the Hilbert space K of all sequences

$$x = \{ x_i \}_{-\infty}^{\infty} \quad , \sum_{-\infty}^{\infty} \| x_i\|^2 < \infty \quad . \text{ If } y = \{ y_i \} \text{, then the}$$

scalar product $< x , y >$ is defined as

$$< x , y > = \sum_{-\infty}^{\infty} < x_i , y_i >$$

we consider a bounded sequence of operators on H , $\{ P_n \}$ and define theoperator P on K by the formula

$$(P x)_n = P_n x_n$$

and the operator U on K by

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$$(Ux)_n = x_{n+1}$$

It is easy to see that we have the relations

$$(U^*x)_n = x_{n+1}, \qquad (P^*x)_n = P_n X_n$$

and thus the operator $T = UP$ has the property

$$(T^*Tx)_n = P_n^2 x_n \qquad (TT^*x)_n = P_{n-1}^2 x_n$$

From these we obtain that T is hyponormal if and only if

$$P_n^2 \geqslant P_{n-1}^2$$

for all n. Also the operator T^2 is hyponormal if and only if

$$P_n P_{n+1}^2 P_n - P_{n-1} P_n^2 P_n \geqslant 0$$

for all n. Now if $H = \ell^2$ and the operators P_n are defined in the following manner, we obtain the desired operator . Let

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and clearly $D - C$ is positive. we set

$$P_n = \begin{cases} C & \text{if } n \leq 0 \\ D^{1/2} & \text{if } n > 0 \end{cases}$$

and in this case $P_n^2 - P_{n-1}^2 \geqslant 0$ is satisfied for all n. But

$$P_0 P_{-1}^2 P_0 = C^2, \qquad P_1 P_2^2 P_1 = D^2$$

and $D^2 - C^2$ is not positive. Thus T is hyponormal and T^2 is not .

We prove now that any operator of class (N) is of class (N, K) .Indeed , we prove the assertion that if T is of class (N) and of class (N , K) , then it is of class (N, k + 1) . It is obvious that this implies our assertion. We have

$$\| T^{k+1} x \| = \left\| \frac{T^k Tx}{\| Tx \|} \right\| \|Tx\| > \|T^2 x\|^k \| Tx \|^{1-k}$$

$$\geqslant \| Tx \|^{2k} \|Tx\|^{1-k}$$

$$= \|Tx\|^{k+1}$$

In the case Tx = 0, the assertion is obvious .We prove now that each operator of class (N, k) is normaloid . Indeed, we can assume, without lossof generality, that $|| T || = 1$ and let $\{ x_n \}$, $|| x_n|| = 1$, such that $|| Tx_n|| \to 1$.

From the definition it is clear that

$$\lim_{n\to\infty} || T^k x_n|| = 1$$

and from this follows that for all integers $i \in [2, k]$.

$$\lim_{n\to\infty} || T^i x_n|| = 1$$

Now from

$$|| T^{k+1}x_n||= \left\|\frac{T^k T X_n}{TX_n}\right\| \quad \frac{1}{\|TX_n\|} \geq \frac{\|T^2X_n\|^2}{\|TX_n\|^{k-1}} \to 1$$

and thus, $\lim_{n\to\infty} || T^{k+1}x_n|| = 1$

An induction argument proves the fact that for all m

$$\lim_{n\to\infty} || T^m x_n|| = 1$$

and thus $|| T^m|| = 1$. The assertion is proved.

We give now an example of an operator which is normaloid and not in theclass (N, K) for all k. Forth is we consider a nilpotent operator T_1 on a Hilbert space H_1 and with the property that $|| T_1|| = 1$. Let T_2 be a normal operator on a Hilbert space H_2 such that $|| T_2|| =1$. Clearly the operator $T = T_1 \oplus T_2$ on $H_1 \oplus H_2$is normaloid and is not in the class (N, K) for all k sinceT_1 is nilpotent .

Hence our main result is proved.

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