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Analytical Study of Linear Operators and its Consequences

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***Abstract*--This paper presents the study of important classes of Linear operators on Hilbert Space including projections. Here, we discuss uses of Riesz Representation Theorem which characterizes Linear functional and observable of a system represented by a space "A" of Linear operators on a Hilbert space in its stability in Quantum Mechanical System with the property of positivity and Normalization. Here, it is proved in this paper that the theory of Linear operators find its consequences in various problems of mathematical physics and Applied Mathematics.**

***Keyword*-- Hilbert space, Reisz Representation, Duality of Hilbert space, Orthogonality of Projection, Normalization and positivity, Direct sum.**

I. INTRODUCTION

Kothe (3, 4) , is the pioneer worker of the present area . In fact, the present work is the extension of work done by Wong, Yau – Chuen (9), Ghosh et al. (01), Ghosh et al. (02), Prasad et al. (05), Srivastava et al. (6), Srivastava et al . (7), and Srivastava et al. (8). In this paper we have studied a new Characterization of Linear operators and its Stability.

Here, we use the following definitions, Notations and Fundamental Ideas:

If M and N are subspaces of a Linear space X such that every $x \in X$ *can* be written uniquely as $x = y + z$ where $y \in M$ & $z \in N$ then the direct sum of M and N can also be written $X = M \oplus N$ where N is called complimentary subspace of M in X and if $M \cap N = \{0\}$, the decomposition $x = y + z$ is unique. A given subspace M has many complimentary subspaces and every complimentary subspace of M has the same dimension and the dimension of a complimentary subspace is called co-dimension of M in X , as if $X = R^3$ and M is a plane through the origin then any line through the origin that does not lie in M is a complimentary subspace.

If $X = M \oplus N$ then we define the projection $P: X \to X$ of X on to M along N by $Px = y$, where $x = y+z$ with $y \in M$, $Z \in N$ which is Linear with ran P = M and ker P = N satisfying $\mathbf{P^2 = P}$. This property characterizes projections for which the following definitions and theorems follow : -

Definition 1: Any projection associated with a direct sum decomposition of a projection on a Linear space X is a linear map $P:X \to X$ such that $P^2 = P$

Definition 2: An orthogonal projection on a Hilbert space H is also a Linear mapping $P:H \to H$ satisfying $P^2 = P$, $\langle Px,y\rangle = \langle x , Py\rangle$ for all $x, y \in H$.

"An orthogonal projection is necessarily bounded."

Theorem 1 : Let X be a linear space,

(i) If $P:X \to X$ is a projection then $X = \text{ran } P \oplus \ker P$
(ii) If $X = M \oplus N$ where M and N are Linear subspaces of X then there is a projection
$P:X \to X$ with ran P = M and ker P = N.

Proof:

For (i) We show that $x \in \text{ran } P$ if $x = Px$
If $x = Px$ then clearly $x \in \text{ran } P$
If $x \in \text{ran } P$ then $x = Py$ for some $y \in x$
and since $P^2 = P$ which follows that $Px = P^2y = Py = x$
If $x \in \text{ran } P \cap \ker P$ then $x = Px$ & $Px = 0$ So $\text{ran } P \cap \ker P = \{0\}$. If $x \in X$ then
We have $x = Px + (x - Px)$; where $Px \in \text{ran } P$ and $(x - Px) \in \ker P$.
Since $P (x - Px) = Px - P^2x = Px - Px = 0$

Thus $X = \text{ran } P \oplus \ker P$. ………………………(1.1)

Now for (ii)

We consider if $X = M \oplus N$ then $x \in N$ has unique decomposition $x = y+z$ with
$y \in M$ & $Z \in N$ and $Px = y$ defines the required Projection .

In particular, in orthogonal subspaces while using Hilbert Space, let us
suppose that M is a closed subspace of Hilbert Space H then by well known property we have $H = M \oplus M^{\perp}$. We call the projection of H on to M along $M^{\perp}$ the orthogonal projection of H on to M.

If $x = y+z$ and $x_1 = y_1 + z_1$ where $y, y_1 \in M$ and $z, z_1 \in M^\perp$ then by orthogonality of M and

$M^\perp \Rightarrow \langle Px, x_1\rangle = \langle y, y_1 + z_1\rangle = \langle y, y_1\rangle = \langle y+z, y_1\rangle$

$= \langle x, Px_1\rangle$ …………… (1.2)

Which states that an orthogonal projection is self Adjoint. We show the properties (1.1) and (1.2) characterize orthogonal projections with Defn-2 .

Lemma :- If P is a non zero orthogonal projection then $\| P\| = 1$.

Proof : - If $x \in H$ and $Px \neq 0$ then by Cauchy Schwarz inequality ,

$$\| Px \| = \frac{\langle Px, Px\rangle}{\| Px \|} = \frac{\langle x, P^2x \rangle}{\| Px \|} = \frac{\langle x, Px \rangle}{\| Px \|} \le \| x\|$$

Therefore $\| P \| \le 1$. If $P \neq 0$ then there is an $x \in H$ with $Px \neq 0$ and $\| P(Px) \| = \| Px \|$ so that $\| P\| \ge 1$.

Thus, the Orthogonal Projection P and closed subspace M of H such that ran P = M will must obey one –one correspondence, then the kernel of Orthogonal Projection is the Orthogonal Complement of M .

Theorem.2 : - Let H be a Hilbert Space .

(i) If P is an Orthogonal projection on H, then P is closed and H = ran P ⊕ ker P is orthogonal direct sum of ran P and kerP.

(ii) If M is a closed subspace of H, then there is an Orthogonal Projection P on H with ran P = M and ker P. $= M^\perp$.

Proof: - For (i), Let us consider P is an orthogonal Projection on H then by the

Theorem . 1, we have μ = ran P + ker P

If $x = Py \in$ ran P and $z \in$ ker P, then $\langle x, z\rangle = \langle Py, z\rangle = \langle y, Pz \rangle = 0$ so

ran P⊕ ker P. Hence, we observe that H is the Orthogonal direct sum of ran P and ker P which follows that ran P = $(\ker P)^\perp$, so ran P is closed.

For (ii), Suppose that M is a closed subspace of H, then by well known property we have

$H = M \oplus M^\perp$

Now we define a Projection $P : H \to H$ by $Px = y$ where $x = y+z$ with $y \in M$ and $z \in M^\perp$, then ran P = M and ker P = $M^\perp$, the orthogonality of P shown in theorem -1.

If P is an orthogonal Projection on H with range M and associated direct sum $H= M \oplus N$ then I-P is the Orthogonal Projection with range N and associated with Orthogonal direct sum $H = N \oplus M$.

Which completes the proof of theorem .2

Example .1 – The space L^2 (R) is the Orthogonal direct sum of space M of even functions and the space N of odd functions .

The Orthogonal Projection P and Q of H onto M and N, respectively are given by

$$\text{by } Pf(x) = \frac{f(x) + f(-x)}{2}, \quad Q f(x) = \frac{f(x) - f(-x)}{2}$$

Where **I- P = Q.**

Example 2 – If $\mathbf{H = R^n}$, the orthogonal projection **Pu** in the direction of a unit vector **u** has the rank one matrix Uu^T. The component of a vector X in the direction of U is

$$\mathbf{PuX = (u^T X)\, u}$$

Example 3 : - If $\mathbf{H = L^2\ (T)}$ is the space of 2π- Periodic function and $\mathbf{u = 1/\sqrt{2\pi}}$ is the constant function with norm one, then the Orthogonal projection P_u maps a function to its mean :

Pu $\mathbf{f = \langle f \rangle}$

$$\textbf{Where } \langle f \rangle = 1/2\pi \int_0^{2\pi} f(x)\, dx$$

The corresponding Orthogonal decomposition,

$\mathbf{f(x) = \langle f \rangle + f'(x)}$ decompose a function in to a constant mean part $\mathbf{\langle f \rangle}$ and a fluctuating part $\mathbf{f'}$ with zero mean .

Example: 4 Suppose $\mathbf{H = L^2\ (T)}$, then for each $\mathbf{n \in Z}$ the functional

$\mathbf{\varphi_n : L^2(T) \to C,\ \ \varphi_n (f) = 1/\sqrt{2\pi} \int_T f(x)\, e^{-inx}\, dx}$ that maps a function to its nth Fourier coefficient is a bounded linear functional. We also have $\mathbf{\| \varphi_n \| = 1}$ for every $\mathbf{n \in Z}$ **.**

Proposition : (a) A Linear functional on a Complex Hilbert space H is a Linear map from H to C. A Linear functional φ is bounded or continuous, if there exists a constant M such that $| \varphi (x) | \le M \| x \|$ for all $x \in H$.

The norm of bounded linear functional φ is

$$\| \varphi \| = \sup_{\| x \| = 1} |\varphi (x)|$$

If $y \in H$ then $\varphi_y (x) = \langle y, x \rangle$ is a bounded Linear functional on H, with $\| \varphi_y \| = \| y \|$.

(b) If φ is a bounded Linear functional on a Hilbert space H, then there is a unique vector $y \in H$ such that

$$\varphi (x) = \langle y, x \rangle \text{ for all } x \in H$$

Thus, from above definitions, theorems, Lemma, examples & propositions (a) & (b) which shows duality of Hilbert space and Riesz representation have the main Result as follows : -

II. Main Result

In Quantum Mechanics, the observable of a system are represented by a space "A " of Linear operators on Hilbert space H. **A** state $\boldsymbol{w}$ of a quantum mechanical system is a linear functional $\boldsymbol{w}$ on the spaces A of observables with the following two properties .

(i) $\boldsymbol{w}\,(A^*\mathbf{A}) \geq 0$ for all $\mathbf{A} \in A$

(ii) $\boldsymbol{w}\,(I) = 1$

Where $\boldsymbol{w}$ (A) is the expected value of the observable A when the system is in the state $\boldsymbol{w}$. Condition (i) is called positivity and condition (ii) is called normalization .

III. Proof Of The Main Result

Suppose that $H = C^n$ and A is the space of all n x n complex matrices, Then A is a Hilbert space with the inner product given by

$$\langle A, B \rangle = \text{tr } A^*B$$

Now, by the Riesz representation theorem for each state $\boldsymbol{w}$ there is a unique $\rho \in A$ such that $\boldsymbol{w}\,(A) = \text{tr } \rho^* A$ for all $A \in A$ and then by conditions of positivity and normalization translate into

$\rho \geq 0$ and $\text{tr } \rho = 1$ respectively.

Hence Proved .

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