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Study on Static Perfect Fluid Spheres in Einstein-Cartan Theory

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***Abstract*-- This paper presents the study of Static perfect Fluid Spheres in Einstein – cartan theory. Here, we consider E – C field equations for static fluid spheres by adopting Tolman's technique. Here, it is proved in this paper that the field equations can be solved in two cases by calculating pressure and density for the distribution.**

***Keyword*-- Static Fluid Spheres, Tolman's Technique, Field Equations, Einstein – Cartan Theory, Gravitational Field.**

I. Introduction

Arkuszewski, W.(1), and Banerjee, S. (2) and Prasanna, A.R.(6) are the pioneer workers of the present area. In fact, the present work is the extension of work done by Bowers, R.L. and Liang, E.P.T. (3), Cartan, E (4, 5), Ray Chaudhary, A.K. et. al. (7), Singh, T. et. al. (8), Tolman, R.C. (9), and Yadav, et. al. (10 & 11). In this paper, we have studied Static Perfect Fluid Spheres in Einstein – Certan Theory.

Here, we use the following Fundamental Ideas:

In recent years much interest has been focused in Einstein-Cartan theory. As a matter of fact the general theory of relativity which has been considered as "most beautiful creation of single mind" has enjoyed success wherever a test has been possible.

The general theory of relativity has also led under general considerations to the existence of singularities in the universe. Since the singularity is not a desirable feature for any physical theory, is it possible to keep this beautiful theory unmolested with regard to its success but at the same time modify it so as to prevent singularities. The answer seems to be in affirmative if one considers the most natural generalization of Einstein's theory as originally suggested by Cartan which is now known as Einstein-Cartan theory (or Einstein-Cartan theory). In this theory the intrinsic spin of matter is incorporated as the source of torsion of the space-time manifold. According to the relativistic quantum mechanics mass and spin are two fundamental characters of an elementary particle system. The energy momentum is source of curvature. By introducing torsion and relating it to the density of intrinsic angular momentum the Einstein-Cartan theory restores the analogy between mass and spin which extends to the principle of equivalence at least in its weak form. According to this principle the world line of a spin less test particle moving under the influence of gravitational fields only depends on its initial position and velocity but not on its mass.

II. Mathematical Treatment Of The Problem

1. The Field Equations

The Einstein-Cartan field equations are

$$R^i_j - \frac{1}{2} R\delta^i_j = -\mathrm{k}t^i_j, \qquad (1)$$

$$Q^i_{jk} - \delta^i_j Q^l_{lk} - \delta^i_k Q^l_{jl} = -kS^i_{jk} \qquad (2)$$

where Q^i_{jk} is torsion tensor, t^i_j is the canonical asymmetric energy momentum tensor S^k_{ij} is the spin tensor, k = –8π

Here we take a static spherically symmetric matter distribution represented by the line element.

$$ds^2 = -e^{2\alpha} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + e^{2\beta} dt^2 \qquad (3)$$

where α & β are functions of r alone. If θ′ represents an orthonormal conframe, than we have

$$\theta^1 = e^{\alpha}\, dr, \theta^2 = rd\,\theta, \theta^3 = r\sin\theta\, d\,\phi, \theta^4 = e^{\beta} dt. \qquad (4)$$

The metric (3) now becomes

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$$ds^2 = -\left\{(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 - (\theta^4)^2\right\}$$

so that

$$g_{ij} = diag \quad \{-1,-1,-1,1\}$$

Assuming that the spins of the individual particles composing the fluid are all aligned in the radical direction we get for the spin tensor S_{ij} the only independent non-zero component to be $S_{23} = K$, say. Since the fluid is supposed to be static, we have the velocity

Four-vector $u^i = \delta_4^{\ i}$.

Thus the non-zero components of S^i_{jk} are

$$S^4_{23} = -S^4_{32} = K\,. \qquad (5)$$

Hence from the Cartan equation (2), we get for Q^i_{jk} the components

$$Q^4_{23} = -Q^4_{32} = -kK \qquad (6)$$

and the others are zero

Using (6) in (3) we can obtain the torsion two-form (H)′ to be

$$(H)^1 = 0,\ (H)^2 = 0, (H)^3 = 0, (H)^4 = -kK\,\theta^2 \wedge \theta^3\,. \qquad (7)$$

Once we have the torsion form we can use it in (3) along with (4) and solve the components of ω^k_l, which in the present case turn out to be

$$\omega^1_4 = \omega^4_1 = e^{-\alpha}\,\beta'\,\theta^4, \omega^2_1 = -\omega^1_2 = \frac{e^{-\alpha}}{r}\theta^2, \qquad (8)$$

$$\omega^2_4 = \omega^4_2 = -\frac{1}{2}kK\theta^3, \omega^3_1 = -\omega^1_3 = \frac{e^{-\alpha}}{r}\theta^3\,,$$

$$\omega^3_4 = \omega^4_3 = -\frac{1}{2}kK\theta^2, \omega^3_2 = -\omega^2_3 = -\frac{1}{2}kK\theta^4 + \frac{\cot\theta}{r}\theta^3\,.$$

Using (8) in (4), we get the curvature form Ω^k_l to be

$$\Omega^1_4 = \left[e^{-2\alpha}\left(\beta'' + \beta'^2 - \alpha'\beta'\right)\right]\left(\theta^1 \wedge \theta^4\right) - \frac{kK}{r}e^{-\alpha}\left(\theta^2 \wedge \theta^3\right), \qquad (9)$$

$$\Omega^3_4 = -\frac{1}{2}ke^{-\alpha}\left[K' + \frac{K}{r}\right]\left(\theta^1 \wedge \theta^2\right) + \left[\frac{e^{-2\alpha}}{r}\beta' + \frac{1}{4}k^2K^2\right]\left(\theta^3 \wedge \theta^4\right),$$

$$\Omega_2^1 = \frac{e^{-2\alpha}}{r}\alpha'\left(\theta^1 \wedge \theta^2\right) - \frac{1}{2}kKe^{-\alpha}\left[\beta' - \frac{1}{r}\right]\left(\theta^4 \wedge \theta^3\right),$$

$$\Omega_3^1 = \frac{e^{-2\alpha}}{r}\alpha'\left(\theta^1 \wedge \theta^3\right) - \frac{1}{2}kKe^{-\alpha}\left[\beta' - \frac{1}{r}\right]\left(\theta^4 \wedge \theta^2\right),$$

$$\Omega_3^2 = \left(\frac{1\text{-}e^{-2\alpha}}{r^2} + \frac{1}{4}k^2K^2\right)\left(\theta^2 \wedge \theta^3\right) + \frac{1}{2}ke^{-\alpha}\left[K' + K\beta'\right]\left(\theta^1 \wedge \theta^4\right).$$

Equations (4) and (9) together give

$$R_{1414}^1 = e^{-2\alpha}\left(\beta'' + \beta'^2 - \alpha'\beta'\right), \tag{10}$$

$$R_{424}^2 = R_{434}^3 = \frac{1}{4}k^2K^2 + \frac{e^{-2\alpha}\beta'}{r},$$

$$R_{212}^1 = R_{313}^1 = \frac{e^{-2\alpha}\,\alpha'}{r},$$

$$R_{323}^2 = \frac{1-e^{-2\alpha}}{r^2} + \frac{1}{4}k^2K^2,$$

$$R_{423}^1 = -\frac{kK^2}{r}e^{-\alpha},$$

$$R_{413}^2 = -R_{412}^3 = \frac{1}{2}ke^{-\alpha}\left(K' + \frac{K}{r}\right),$$

$$R_{243}^1 = -R_{342}^1 = \frac{1}{2}kKe^{-\alpha}\left(\beta' - \frac{1}{r}\right),$$

$$R_{314}^2 = \frac{1}{2}e^{-\alpha}\left(K' + K\beta'\right).$$

The Ricci tensor R_{ij} and scalar of curvature R are therefore given by

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$$R_{11} = -e^{-2\alpha}\left(\beta'' + \beta'^2 - \alpha'\beta' - \frac{2\alpha'}{r}\right), \qquad (11)$$

$$R_{22} = R_{33} = -\frac{e^{-2\alpha}}{r^2}\left[1 + r\left(\beta' - \alpha'\right)\right] + \frac{1}{r^2},$$

$$R_{44} = e^{-2\alpha}\left(\beta'' + \beta'^2 - \alpha'\beta' + \frac{2\beta'}{r}\right) + \frac{1}{2}k^2K^2,$$

$$R = -2\left\{\frac{1}{r^2} - e^{-2\alpha}\right\}\left[\frac{1}{r^2} + \beta'' + \beta'^2 - \alpha'\beta' + \frac{2}{r}\left(\beta' - \alpha'\right)\right] + \frac{1}{2}kK^2 \qquad (12)$$

with $R_{ij} = 0$, $i \neq j$. Hence the Einstein tensor $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ji}$ is found to have the non-zero components.

$$G_{11} = -\frac{1}{r^2} + e^{-2\alpha}\left(\frac{2\beta'}{r} + \frac{1}{r^2}\right) + \frac{1}{4}k^2K^2, \qquad (13)$$

$$G_{22} = G_{33} = e^{-2\alpha}\left[\beta'' + \beta'^2 - \alpha'\beta' + \frac{1}{r}(\beta' - \alpha') + \frac{1}{4}k^2K^2\right],$$

$$G_{44} = \frac{1}{r^2} + e^{-2\alpha}\left(\frac{2\alpha'}{r} - \frac{1}{r^2}\right) + \frac{1}{4}k^2K^2.$$

Since we are considering a perfect fluid distribution with isotropic pressure p and matter density ρ, we have

$$t_i^j = R^{jk}\left\{\left[(\rho + p)u_k - u^l\,\nabla_m\left(u^m S_{lk}\right)\right]u_k - pe_{ki}\right\}. \qquad (14)$$

Using (6) we get then the non-zero components

$$t_1^1 = t_2^2 = t_3^3 = -p . t_4^4 = \rho . \qquad (15)$$

Hence, using (13) and (15), the field equations (1) may be written as

$$-\frac{1}{r^2} + e^{-2\alpha}\left(\frac{2\beta'}{r} + \frac{1}{r^2}\right) + \frac{1}{4}k^2K^2 = -k\,p, \qquad (16)$$

$$e^{-2\alpha}\left[\beta'' + \beta'^2 - \alpha'\beta' + \frac{1}{r}(\beta' - \alpha')\right] + \frac{1}{4}k^2K^2 = -k\rho, \qquad (17)$$

$$-\frac{1}{r^2}-e^{-2\alpha}\left(\frac{2\alpha'}{r}-\frac{1}{r^2}\right)-\frac{1}{4}k^2K^2=k\rho. \qquad (18)$$

The conservation laws give us the relations

$$\nabla.\left[(\rho+p)u'\right]=0 \text{ (matter conservation),} \qquad (19)$$

$$\nabla.(Ku')=0 \text{ (spin conservation) and ,} \qquad (20)$$

$$\frac{dp}{dr}+(\rho+p)\beta'+\frac{1}{2}kK(K'+K\beta')=0 \quad . \qquad (21)$$

If we assume the equation of hydrostatic equilibrium to hold as in general relativity, namely

$$\frac{dp}{dr}+(\rho+p)\beta'=0. \qquad (22)$$

We get the additional equation

$$K'+K\beta'=O. \qquad (23)$$

Solving for K we get

$$K=He^{-\beta} \qquad (24)$$

where H is a constant of integrations to be determined. Setting

$k=-\dfrac{8\pi G}{c^2}$ with C = 1, c = 1. From the field equations, we can obtain

$$8\pi p=16\pi^2K^2-\frac{1}{r^2}+e^{-2\alpha}\left(\frac{2\beta'}{r}+\frac{1}{r^2}\right) , \qquad (25)$$

$$8\pi\rho=16\pi^2K^2+\frac{1}{r^2}+e^{-2\alpha}\left(\frac{2\alpha'}{r}+\frac{1}{r^2}\right), \qquad (26)$$

$$e^{-2\alpha}\left[\left(\frac{\beta''}{r}-\frac{\beta'}{r^2}-\frac{1}{r^3}\right)-\alpha'\left(\frac{2\beta'}{r}+\frac{1}{r^2}\right)+\beta'\left(\frac{\alpha'+\beta'}{r}\right)\right]+\frac{1}{r^3}=0 \qquad (27)$$

where K = He$^{-\beta}$ where H is a constant

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In principle, we now have a completely determined system if an equation of state is specified. However, it is well known that in practice this set of equations is formidable to solve using a reassigned equation of state, except perhaps for the case ρ = p, which may not be physically meaningful.

Secondly, we have the question of boundary conditions to be chosen for fitting the solutions in the interior and the exterior of the state. A very interesting aspect of the Einstein-Cartan theory is that outside the fluid distribution the equations reduce to Einstein's equations for empty space, viz., $R_{ij} = 0$, since there is no spin density.

Following Hehl's approach, if we define

$$\bar{p} = p - 2\pi K^2, \bar{\rho} - 2\pi K^2 \qquad (28)$$

then we find that the equations take the usual general relativistic form for a static fluid sphere as given by

$$8\pi \bar{p} = -\frac{1}{r^2} + e^{-2\alpha}\left(\frac{2\beta'}{r} + \frac{1}{r^2}\right), \qquad (29)$$

$$8\pi \bar{\rho} = -\frac{1}{r^2} + e^{-2\alpha}\left(\frac{2\beta'}{r} - \frac{1}{r^2}\right) \qquad (30)$$

with (27) remaining the same. The equation of continuity (21) now becomes

$$\frac{d\bar{p}}{dr} + (\bar{\rho} + \bar{p})\beta' = 0 . \qquad (31)$$

It is clear from these equations that it is the $\bar{p}$ and not the p which is continuous across the boundary *r* = a, of the fluid sphere. The continuity of $\bar{p}$ across the boundary ensures that of β' (exp.2β). Further with $\bar{p}$ and $\bar{\rho}$ replacing p and ρ respectively we are assured that the metric coefficients are continuous across the boundary. Hence we shall apply the usual boundary conditions to the solution of equations (27), (29) and (30). We use the boundary conditions

$$\left[e^{-2\alpha}\right]_{r=a} = \left[e^{2\beta}\right]_{r=a} = \left(1 - \frac{2m}{a}\right), \qquad (32)$$

and

$$\bar{p} = 0 \ at \ r = a \qquad (33)$$

where a is the radius of the fluid sphere and m is the mass of the fluid sphere. The total mass as observed by an external observer, inside the fluid sphere of radius a is given by

$$m = \int_0^a \rho\, r^2 dr = 4\pi \int_0^a \rho\, r^2 dr - 8\pi^2 \int_0^a K^2(r) r^2 dr . \qquad (34)$$

Thus, the total mass of the fluid sphere is modified by the correction

$$8\pi^2 \int_0^a K^2(r) r^2\, dr .$$

Equations (29), (30) and (31) are the same as obtained by Tolman [145], so we can use the same solutions for our discussion. Assuming that the sphere has a finite radius r = a for r > a, since the equations are $R_{ij} = 0$, we have by Birkhoff's theorem the space-time metric represented by the Schwarzschild solution

$$ds^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\,\theta^2 - r^2 \sin^2\theta\, d\,\phi^2 + \left(1-\frac{2m}{r}\right)dt^2 \qquad (35)$$

where m is a constant associated with the mass of the sphere.

III. Solutions

Case (I): Here we assume

$$e^{2\beta} = \lambda r^6 \qquad (36)$$

where λ is constant.

Then solving equation (27) for α we get

$$e^{2\alpha} = \frac{-2}{1-\nu r^{-1}}\,. \qquad (37)$$

where *v* is constant.

Thus, the line element is given by

$$ds^2 = \frac{1}{1-2\nu\, r^{-1}} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + \lambda r^6\, dt^2\,. \qquad (38)$$

Using boundary conditions the constants are found to be

$$\lambda = a^{-6}\left(1-\frac{2m}{a}\right), \nu = \frac{4m-3a}{a^2}\,. \qquad (39)$$

The pressure and density are evaluated to be

$$8\pi\, p = \frac{16\pi^2 H^2}{\lambda r^6} - \frac{1}{r^2} - \frac{7}{2r^3}(r+2\nu)\,, \qquad (40)$$

$$8\pi\, \rho = \frac{16\pi^2 H^2}{\lambda r^6} + \frac{1}{r^2} + \frac{(2\nu - r)^2}{r^4}\,. \qquad (41)$$

Case (II) : Here we assume

$$e^{2\beta} = cr \qquad (42)$$

where c is a constant.

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Using (42) in (27), after calculations we get α finally as

$$e^{2\alpha} = \frac{7}{4+7Br^{7/3}} \qquad (43)$$

where B is a constant.

Thus the line element is given by

$$ds^2 = cr\, dt^2 - \frac{7}{4+7Br^{7/3}} dr^2 - r^2(d\,\theta^2 + \sin^2\theta\, d\phi^2), \qquad (44)$$

$$C = \frac{1}{2a}, B = -\frac{a^{-7/3}}{14} \qquad (45)$$

The pressure and density are given by

$$8\pi\, p = \frac{16\pi^2 H^2}{cr} + \frac{1}{7r^2}\left\{1 - \left(\frac{r}{a}\right)^{7/3}\right\}, \qquad (46)$$

$$8\pi\, \rho = \frac{16\pi^2 H^2}{cr} + \frac{3}{7r^2}\left\{1 + \frac{5}{9}\left(\frac{r}{a}\right)^{7/3}\right\}. \qquad (47)$$

The spin density K is given by

$$K^2 = \frac{2H^2 a}{r}\ . \qquad (48)$$

IV. Conclusion

We conclude that the predictions of Einstein-Cartan theory differ from those of general relativity only for matter filled regions, therefore besides cosmology, an important application field of Einstein-Cartan theory is relativistic astrophysics which deals with the theories of stellar objects like neutron stars with some alignment of spins of the constituent particles. Hence it is desirable to understand the full implication of the Einstein-Cartan theory for finite distributions like fluid spheres with non-zero pressure.

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